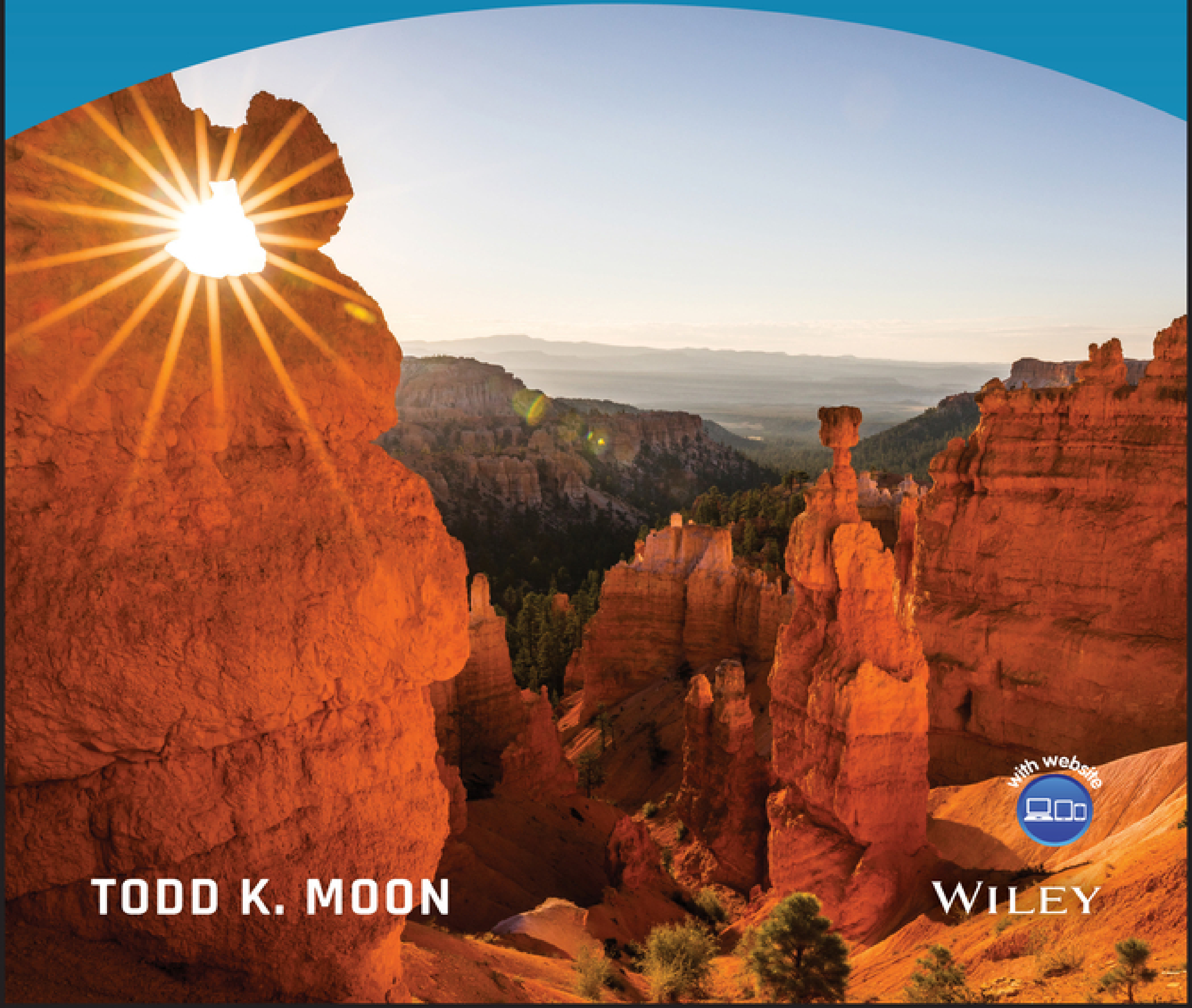
ERROR CORRECTION CODING

MATHEMATICAL METHODS AND ALGORITHMS

FIRST EDITION



TODD K. MOON

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Second Edition

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Preface

The goals of this second edition are much the same as the first edition: to provide a comprehensive introduction to error correction coding suitable for the engineering practitioner and to provide a solid foundation leading to more advanced treatments or research. Since the first edition, the content has been modernized to include substantially more on LDPC code design and decoder algorithms, as well as substantial coverage of polar codes. The thorough introduction to finite fields and algebraic codes of the first edition has been retained, with some additions in finite geometries used for LDPC code design. As observed in the first edition, the sophistication of the mathematical tools used increases over time. While keeping with the sense that this is the first time most readers will have seen these tools, a somewhat higher degree of sophistication is needed in some places.

The presentation is intended to provide a background useful both to engineers, who need to understand algorithmic aspects for the deployment and implementation of error correction coding, and to researchers, who need sufficient background to prepare them to read, understand, and ultimately contribute to the research literature. The practical algorithmic aspects are built upon a firm foundation of mathematics, which are carefully motivated and developed.

Pedagogical Features

Since its inception, coding theory has drawn from richly interacting variety of mathematical areas, including detection theory, information theory, linear algebra, finite geometries, combinatorics, optimization, system theory, probability, algebraic geometry, graph theory, statistical designs, Boolean functions, number theory, and modern algebra. The level of sophistication has increased over time: algebra has progressed from vector spaces to modules; practice has moved from polynomial interpolation to rational interpolation; Viterbi makes way for SOVA and BCJR. This richness can be bewildering to students, particularly engineering students who may be unaccustomed to posing problems and thinking abstractly. It is important, therefore, to motivate the mathematics carefully.

Some of the major pedagogical features of the book are as follows.

- While most engineering-oriented error-correction-coding textbooks clump the major mathematical concepts into a single chapter, in this book the concepts are developed over several chapters so they can be put to more immediate use. I have attempted to present the mathematics "just in time," when they are needed and well-motivated. Groups and linear algebra suffice to describe linear block codes. Cyclic codes motivate polynomial rings. The design of cyclic codes motivates finite fields and associated number-theoretical tools. By interspersing the mathematical concepts with applications, a deeper and broader understanding is possible.
- For most engineering students, finite fields, the Berlekamp–Massey algorithm, the Viterbi algorithm, BCJR, and other aspects of coding theory are abstract and subtle. Software implementations of the algorithms bring these abstractions closer to a meaningful reality, bringing deeper understanding than is possible by simply working homework problems and taking tests. Even when students grasp the concepts well enough to do homework on paper, these programs provide a further emphasis, as well as tools to *help* with the homework. The understanding becomes *experiential*, more than merely conceptual.

  Understanding of any subject typically improves when the student him- or herself has the chance to teach the material to someone (or something) else. A student must develop an especially clear understanding of a concept in order to "teach" it to something as dim-witted and literal-minded as a computer. In this process the computer can provide feedback to the student through debugging and program testing that reinforces understanding.

  In the coding courses I teach, students implement a variety of encoders and decoders, including Reed–Solomon encoders and decoders, convolutional encoders, turbo code decoders, and LDPC decoders. As a result of these programming activities, students move beyond an on-paper understanding, gaining a perspective of what coding theory can do and how to put it to work. A colleague of mine observed that many students emerge from a first course in coding theory more

LabIntro.pdf

confused than informed. My experience with these programming exercises is that my students are, if anything, overconfident, and feel ready to take on a variety of challenges.

In this book, programming exercises are presented in a series of 13 Laboratory Exercises. These are supported with code providing most of the software "infrastructure," allowing students to focus on the particular algorithm they are implementing.

These labs also help with the coverage of the course material. In my course I am able to offload classroom instruction of some topics for students to read, with the assurance that the students will learn it solidly on their own as they implement it. (The Euclidean algorithm is one of these topics in my course.)

Research in error control coding can benefit from having a flexible library of tools for the computations, particularly since analytical results are frequently not available and simulations are required. The laboratory assignments presented here can form the foundation for a research library, with the added benefit that having written major components, the researcher can easily modify and extend them.

It is in light of these pedagogic features that this book bears the subtitle *Mathematical Methods and Algorithms.*

There is sufficient material in this book for a one- or two-semester course based on the book, even for instructors who prefer to focus less on implementational aspects and the laboratories.

Over 200 programs, functions and data files are associated with the text. The programs are written in MATLAB,[1] C, or C++. Some of these include complete executables which provide "tables" of primitive polynomials (over any prime field), cyclotomic cosets and minimal polynomials, and BCH codes (not just narrow sense), avoiding the need to tabulate this material. Other functions include those used to make plots and compute results in the book. These provide example of how the theory is put into practice. Other functions include those used for the laboratory exercises. The files are highlighted in the book by the icon

as in the marginal note above. The files are available at https://github.com/tkmoon/eccbook. Other aspects of the book include the following:

- Many recent advances in coding have resulted from returning to the perspective of coding as a detection problem. Accordingly, the book starts off with a digital communication framework with a discussion of **detection theory**.
- Recent codes such as polar codes or LDPC codes are capable of achieving capacity, or nearly so. It is important, therefore, to understand what capacity is and what it means to transmit at capacity. Chapter 1 also summarizes **information theory**, to put coding into its historical and modern context. Added in this second edition is an introduction to nonasymptotic information theory, which will be increasingly important in developing communication systems. The information theory informs EXIT chart analysis of turbo and LDPC codes.

[1] MATLAB is a registered trademark of The Mathworks, Inc.

- Pedagogically, Hamming codes are used to set the stage for the book by using them to demonstrate block codes, cyclic codes, trellises and Tanner graphs.
- Homework exercises are drawn from a variety of sources and are at a variety of levels. Some are numerical, testing basic understanding of concepts. Others provide the opportunity to prove or extend results from the text. Others extend concepts or provide new results. Because of the programming laboratories, exercises requiring decoding by hand of given bit sequences are few, since I am of the opinion that is better to know how to tell the computer than to do it by hand. I have drawn these exercises from a variety of sources, including problems that I faced as a student and those which I have given to students on homework and exams over the years.
- Number theoretic concepts such as divisibility, congruence, and the Chinese remainder theorem are developed at a point in the development where students can appreciate it.
- Occasionally connections between the coding theoretic concepts and related topics are pointed out, such as **public key cryptography** and **shift register sequences**. These add spice and motivate students with the understanding that the tools they are learning have broad applicability.
- There has been considerable recent progress made in decoding Reed–Solomon codes by re-examining their original definition. Accordingly, Reed–Solomon codes are defined both in this primordial way (as the image of a polynomial function) and also using a generator polynomial having roots that are consecutive powers of a primitive element. This sets the stage for several decoding algorithms for Reed–Solomon codes, including frequency-domain algorithms, **Welch–Berlekamp algorithm** and the **soft-input Guruswami–Sudan algorithm**.
- **Turbo codes**, including EXIT chart analysis, are presented, with both BCJR and SOVA decoding algorithms. Both probabilistic and likelihood decoding viewpoints are presented.
- **LDPC codes** are presented with an emphasis on the decoding algorithm. Density evolution analysis is also presented.
- **Polar Codes** are a family of codes developed since the first edition. They offer low complexity encode and decode with soft inputs and without iterative decoding. They also provably achieve channel capacity (as the code length goes to infinity). This book offers a deep introduction to polar codes, including careful description of encoding and decoding algorithms, with operational C++ code.
- Several **Applications** involving state-of-the-art systems illustrate how the concepts can be applied.
- **Space-time codes**, used for multi-antenna systems in fading channels, are presented.

Courses of Study

A variety of courses of study are possible. In the one-semester course I teach, I move quickly through principal topics of block, trellis, and iteratively–decoded codes. Here is an outline of one possible one-semester course:

Chapter 1: Major topics only.

Chapter 2: All.

Chapter 3: Major topics.

Chapter 4: Most. Leave CRC codes and LFSR to labs.

Chapter 5: Most. Touch on RSA.

Chapter 6: Most. Light on GFFT and variations.

Chapter 12: Most. Skip puncturing and stack-oriented algorithms.

Chapter 13: Most. Skip the V.34 material.

Chapter 14: Basic definition and the BCJR algorithm.

Chapter 15: Basic definition of LDPC codes and the sum-product decoder.

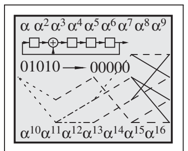
Chapter 17: Introduce the idea of

A guide in selecting material for this course is: follow the labs. To get through all 13 labs, selectivity is necessary.

An alternative two-semester course could be a semester devoted to block codes followed by a semester on trellis and iteratively decoded codes. A two semester sequence could move straight through the book, with possible supplements from the research literature on topics of particular interest to the instructor. A different two-semester approach would follow the outline above for one semester, followed by more advanced coverage of LDPC and polar codes.

Theorems, lemmas, corollaries, examples, and definitions are all numbered sequentially using the same counter in a chapter, which should make identifying these environments straightforward. Figures, tables, and equations each have their own counters. Definitions, examples, and proofs are terminated by the symbol □.

Use of Computers


comptut.pdf

The computer-based labs provide a means of working out some of the computational details that otherwise might require drudgery. These are primarily (with the exception of the first lab) to be done in C++, where the ability to overload operators and run at compiled speed make the language very well suited.

It may be helpful to have a "desk calculator" for homework and exploring ideas. Many tools exist now for this. The brief tutorial `comptut.pdf` provides an introduction to `gap` and `magma`, both of which can be helpful to students doing homework or research in this area. The `sage` package built on Python also provides considerable relevant capability.

Why This Book?

In my mind, the purposes of a textbook are these:

1. To provide a topographical map into the field of study, showing the peaks and the important roads to them. (However, in an area as rich as coding theory, it is impossible to be exhaustive.)
2. To provide specific details about important techniques.
3. To present challenging exercises that will strengthen students' understanding of the material and present new perspectives.
4. To have some reference value, so that practitioners can continue to use it.
5. To provide references to literature that will lead readers to make their own discoveries. With a rapidly-changing field, the references can only provide highlights; web-based searches have changed the nature of the game. Nevertheless, having a starting point in the literature is still important.
6. To present a breadth of ideas which will motivate students to innovate on their own.

A significant difficulty I have faced is selection. The terrain is so richly textured that it cannot be mapped out in a single book. Every communication or information theory conference and every issue of the *IEEE Transactions on Information Theory* yields new and significant results. Publishing restrictions and practicality limit this book from being encyclopedic. My role as author has been merely to select what parts of the map to include and to present them in a pedagogically useful way. In so doing, I have aimed to choose tools for the general practitioner and student (and of interest to me). Other than that selective role, no claim of creation is intended; I hope I have given credit as appropriate where it is due.

This book is a result of teaching a course in error correction coding at Utah State University for nearly three decades. Over that time, I have taught out of the books [45], [489], and [275], and my debt to these books is clear. Parts of some chapters grew out of lecture notes based on these books and the connections will be obvious. I have felt compelled to include many of the exercises from the first coding course I took out of [275]. These books have defined for me the *sine qua non* of error-correction coding texts. I am also indebted to [296] for its rich theoretical treatment, [401] for presentation of trellis coding material, [462] for discussion of bounds, [190] for exhaustive treatment of turbo coding methods, and to the many great researchers and outstanding expositors whose works have contributed to my understanding. More recently, the extensive coverage of LDPC codes in [391] has been extremely helpful, and motivated my own implementation of almost all of the decoding algorithms.

Acknowledgments

I am grateful for the supportive environment at Utah State University that has made it possible to undertake and to complete this task. Students in coding classes over the years have contributed to this material.

I have benefitted tremendously from feedback from careful readers and translators who have provided their own error correction from the first edition. Their negative acknowledgment protocols have improved the delivery of this packet of information. I thank you all!

To my six wonderful children — Leslie, Kyra, Kaylie, Jennie, Kiana, and Spencer — and presently twelve grandchildren, and my wife Barbara, who have seen me slip away too often and too long to write, I express my deep gratitude for their trust and patience. In the end, all I do is for them.

T.K.M
Logan, UT, October 2020

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About the Companion Website

This book is accompanied by a companion website:

www.wiley.com/go/Moon/ErrorCorrectionCoding

The website includes:

- Solutions Manual
- Program Files

Part I

Introduction and Foundations

Chapter 1

A Context for Error Correction Coding

I will make weak things become strong unto them ...

—Ether 12:27

... he denies that any error in the machine is responsible for the so-called errors in the answers. He claims that the Machines are self correcting and that it would violate the fundamental laws of nature for an error to exist in the circuits of relays.

—Isaac Asimov
I, Robot

1.1 Purpose of This Book

Error control coding in the context of digital communication has a history dating back to the middle of the twentieth century. In recent years, the field has been revolutionized by codes which are capable of approaching the theoretical limits of performance, the *channel capacity*. This has been impelled by a trend away from purely combinatoric and discrete approaches to coding theory toward codes which are more closely tied to a physical channel and soft decoding techniques. The purpose of this book is to present error correction and detection coding covering both traditional concepts thoroughly as well as modern developments in soft-decision and iteratively decoded codes and recent decoding algorithms for algebraic codes. An attempt has been made to maintain some degree of balance between the mathematics and their engineering implications by presenting both the mathematical methods used in understanding the codes as well as the algorithms which are used to efficiently encode and decode.

1.2 Introduction: Where Are Codes?

Error correction coding is the means whereby errors which may be introduced into digital data as a result of transmission through a communication channel can be corrected based upon received data. Error detection coding is the means whereby errors can be detected based upon received information. Collectively, error correction and error detection coding are *error control coding*. Error control coding can provide the difference between an operational communications system and a dysfunctional system. It has been a significant enabler in the telecommunications revolution, portable computers, the Internet, digital media, and space exploration. Error control coding is nearly ubiquitous in modern, information-based society. Every compact disc, CD-ROM, or DVD employs codes to protect the data embedded in the plastic disk. Every flash drive (thumb drive) and hard disk drive employs correction coding. Every phone call made over a digital cellular phone employs it. Every frame of digital television is protected by error correction coding. Every packet transmitted over the Internet has a protective coding "wrapper" used to determine if the packet has been received correctly. Even everyday commerce takes advantage of error detection coding, as the following examples illustrate.

Error Correction Coding: Mathematical Methods and Algorithms, Second Edition. Todd K. Moon
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Companion website: www.wiley.com/go/Moon/ErrorCorrectionCoding

Example 1.1 The ISBN (international standard book number) is used to uniquely identify books. An ISBN such as 0-201-36186-8 can be parsed as

$$\underbrace{0}_{\text{country}} - \underbrace{20}_{\text{publisher}} - \underbrace{1 - 36186}_{\text{book no.}} - \underbrace{8}_{\text{check}}.$$

Hyphens do not matter. The first digit indicates a country/language, with 0 for the United States. The next two digits are a publisher code. The next six digits are a publisher-assigned book number. The last digit is a check digit, used to validate if the code is correct using what is known as a weighted code. An ISBN is checked as follows: the cumulative sum of the digits is computed, then the cumulative sum of the cumulative sum is computed. For a valid ISBN, the sum-of-the-sum must be equal to 0, modulo 11. The character X is used for the check digit 10. For this ISBN, we have

	Cumulative	Cumulative
Digit	Sum	Sum
0	0	0
2	2	2
0	2	4
1	3	7
3	6	13
6	12	25
1	13	38
8	21	59
6	27	86
8	35	121

The final sum-of-the-sum is 121, which is equal to 0 modulo 11 (i.e., the remainder after dividing by 11 is 0). □

Example 1.2 The Universal Product Codes (UPC) employed on the bar codes of most merchandise employ a simple error detection system to help ensure reliability in scanning. In this case, the error detection system consists of a simple parity check. A UPC consists of a 12-digit sequence, which can be parsed as

$$\underbrace{0\ 16000}_{\substack{\text{manufacturer}\\ \text{identification}\\ \text{number}}} \quad \underbrace{66610}_{\substack{\text{item}\\ \text{number}}} \quad \underbrace{8}_{\substack{\text{parity}\\ \text{check}}}.$$

Denoting the digits as $u_1, u_2, \ldots, u_{12}$, the parity digit u_{12} is determined such that

$$3(u_1 + u_3 + u_5 + u_7 + u_9 + u_{11}) + (u_2 + u_4 + u_6 + u_8 + u_{10} + u_{12})$$

is a multiple of 10. In this case,

$$3(0 + 6 + 0 + 6 + 6 + 0) + (1 + 0 + 0 + 6 + 1 + 8) = 70.$$

If, when a product is scanned, the parity condition does not work, the operator is flagged so that the object may be re-scanned. □

1.3 The Communications System

Appreciation of the contributions of coding and understanding its limitations require some awareness of information theory and how its major theorems delimit the performance of a digital communication system. Information theory is increasingly relevant to coding theory, because with recent advances in theory it is now possible to achieve the performance bounds of information theory. By contrast, in the past, the bounds were more of a backdrop to the action on the stage of coding research and practice. Part of this success has come by placing the coding problem more fully in its communications context, marrying the coding problem more closely to the signal detection problem instead of treating the coding problem mostly as one of discrete combinatorics.

Information theory treats *information* almost as a physical quantity which can be measured, transformed, stored, and moved from place to place. A fundamental concept of information theory is that information is conveyed by the resolution of uncertainty. Information can be measured by the amount of uncertainty resolved. For example, if a digital source always emits the same value, say 1, then no information is gained by observing that the source has produced, yet again, the output 1, since there was no uncertainty about the outcome to begin with. Probabilities are used to mathematically describe the uncertainty. For a discrete random variable X (i.e., one which produces discrete outputs, such as $X = 0$ or $X = 1$), the information conveyed by observing an outcome x is defined as $-\log_2 P(X = x)$ bits. (If the logarithm is base 2, the units of information are in **bits**. If the natural logarithm is employed, the units of information are in **nats**.) For example, if $P(X = 1) = 1$ (the outcome 1 is certain), then observing $X = 1$ yields $-\log_2(1) = 0$ bits of information. On the other hand, observing $X = 0$ in this case yields $-\log_2(0) = \infty$: total surprise at observing an impossible outcome. The *entropy* is the *average information*. For a binary source X having two outcomes occurring with probabilities p and $1 - p$, the binary entropy function, denoted as either $H_2(X)$ (indicating that it is the entropy of the source) or $H_2(p)$ (indicating that it is a function of the outcome probabilities) is

$$H_2(X) = H_2(p) = E[-\log_2 P(X)] = -p\log_2(p) - (1-p)\log_2(1-p) \text{ bits.}$$

A plot of the binary entropy function as a function of p is shown in Figure 1.1. The peak information of 1 bit occurs when $p = \frac{1}{2}$.

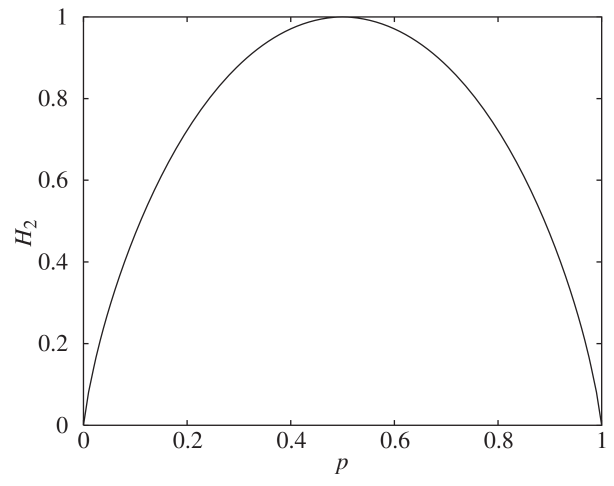

Figure 1.1: The binary entropy function $H_2(p)$.

Example 1.3 A fair coin is tossed once per second, with the outcomes being "head" and "tail" with equal probability. Each toss of the coin generates an event that can be described with $H_2(0.5) = 1$ bit of information. The sequence of tosses produces information at a rate of 1 bit/second. An unfair coin, with $P(\text{head}) = 0.01$ is tossed. The average information generated by each throw in this case is $H_2(0.01) = 0.0808$ bits. Another unfair coin, with $P(\text{head}) = 1$ is tossed. The information generated by each throw in this case is $H_2(1) = 0$ bits. □

For a source X having M outcomes $x_1, x_2, \ldots, x_M$, with probabilities $P(X = x_i) = p_i$, $i = 1, 2, \ldots, M$, the entropy is

$$H(X) = E[-\log_2 P(X)] = -\sum_{i=1}^{M} p_i \log_2 p_i \text{ bits.} \tag{1.1}$$

Note: The "bit" as a measure of entropy (or information content) is different from the "bit" as a measure of storage. For the unfair coin having $P(\text{head}) = 1$, the actual information content determined by a toss of the coin is 0: there is no information gained by observing that the outcome is again 1. For this process with this unfair coin, the entropy rate — that is, the amount of actual information it generates — is 0. However, if the coin outcomes were for some reason to be stored directly, without the benefit of some kind of coding, each outcome would require 1 bit of storage (even though they don't represent any new information).

With the prevalence of computers in our society, we are accustomed to thinking in terms of "bits" — e.g., a file is so many bits long, the register of a computer is so many bits wide. But these are "bits" as a measure of storage size, not "bits" as a measure of actual information content. Because of the confusion between "bit" as a unit of information content and "bit" as an amount of storage, the unit of information content is sometimes — albeit rarely — called a *Shannon*, in homage to the founder of information theory, Claude Shannon.[1]

A digital communication system embodies functionality to perform physical actions on information. Figure 1.2 illustrates a fairly general framework for a single digital communication link. In this link, digital data from a *source* are encoded and modulated (and possibly encrypted) for communication over a *channel*. At the other end of the channel, the data are demodulated, decoded (and possibly decrypted), and sent to a sink. The elements in this link all have mathematical descriptions and theorems from information theory which govern their performance. The diagram indicates the realm of applicability of three major theorems of information theory.

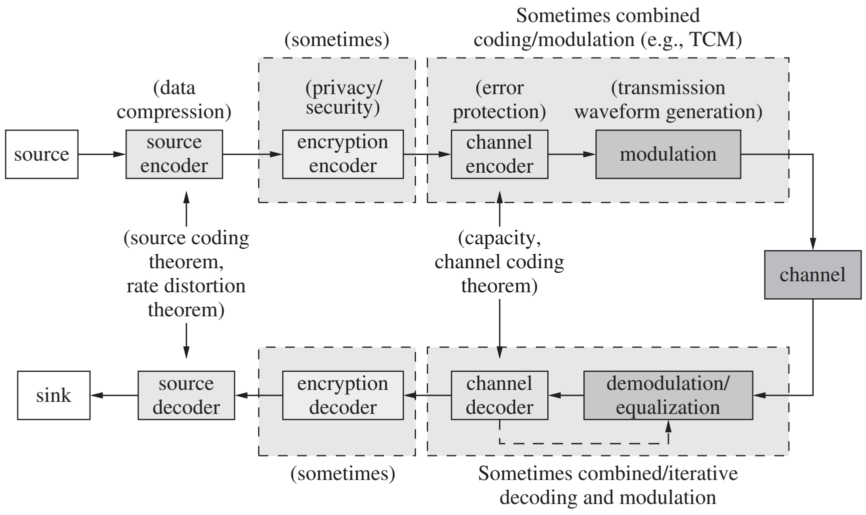

Figure 1.2: A general framework for digital communications.

There are actually many kinds of codes employed in a communication system. In the description below, we point out where some of these codes arise. Throughout the book we make some connections between these codes and our major focus of study, error correction codes.

The source is the data to be communicated, such as a computer file, a video sequence, or a telephone conversation. For our purposes, it is represented in digital form, perhaps as a result of an

[1] This mismatch of object and value is analogous to the physical horse, which may or may not be capable of producing one "horsepower" of power, 550 ft-lbs/second. Thermodynamicists can dodge the issue by using the SI unit of Watts for power, information theorists might sidestep confusion by using the Shannon. Both of these units honor founders of their respective disciplines.

analog-to-digital conversion step. Information-theoretically, sources are viewed as streams of random numbers governed by some probability distribution.

Every source of data has a measure of the information that it represents, which (in principle) can be exactly quantified in terms of entropy.

The source encoder performs data compression by removing redundancy.

As illustrated in Example 1.3, the number of bits used to store the information from a source may exceed the number of bits of actual information content. That is, the number of bits to represent the data may exceed the number of mathematical bits — Shannons — of actual information content.

The amount a particular source of data can be compressed without any loss of information (*lossless* compression) is governed theoretically by the *source coding theorem* of information theory, which states that a source of information can be represented without any loss of information in such a way that the amount of storage required (in bits) is equal to the amount of information content — the entropy — in bits or Shannons. To achieve this lower bound, it may be necessary for long blocks of the data to be jointly encoded.

Example 1.4 For the unfair coin with $P(\text{head}) = 0.01$, the entropy is $H(0.01) = 0.0808$. Therefore, 10,000 such (independent) tosses convey 808 bits (Shannons) of information, so theoretically the information of 10,000 tosses of the coin can be represented exactly using only 808 (physical) bits of information. □

Thus a bit (in a computer register) in principle *can* represents an actual (mathematical) bit of information content, if the source of information is represented correctly.

In compressing a data stream, a source encoder removes redundancy present in the data. For compressed binary data, 0 and 1 occur with equal probability in the compressed data (otherwise, there would be some redundancy which could be exploited to further compress the data). Thus, it is frequently assumed at the channel coder that 0 and 1 occur with equal probability.

The source encoder employs special types of codes to do the data compression, called collectively source codes or data compression codes. Such coding techniques include Huffman coding, run-length coding, arithmetic coding, Lempel–Ziv coding, and combinations of these, all of which fall beyond the scope of this book.

If the data need to be compressed below the entropy rate of the source, then some kind of distortion must occur. This is called lossy data compression. In this case, another theorem governs the representation of the data. It is possible to do lossy compression in a way that minimizes the amount of distortion for a given rate of transmission. The theoretical limits of lossy data compression are established by the *rate–distortion theorem* of information theory. One interesting result of rate–distortion theory says that for a binary source having equiprobable outcomes, the minimal rate to which the data can be compressed with the average distortion per bit equal to p is

$$r = 1 - H_2(p) \quad p \leq \frac{1}{2}. \tag{1.2}$$

Lossy data compression uses its own kinds of codes as well.

Example 1.5 In the previous example, suppose that a channel is available that only provides 800 bits of information to convey 10,000 tosses of the biased coin. Since this number of bits is less than allowed by the source coding theorem, there must be some distortion introduced. From above, the source rate is 0.0808 bits of information per toss. The rate we are sending over this restricted channel is $800/10\ 000 = 0.08$ bits/toss. The rate at which tosses are coded is

$$r = \frac{800}{808} = 0.9901.$$

The distortion is

$$p = H_2^{-1}(1 - r) = H_2^{-1}(1 - 0.9901) = H_2^{-1}(0.0099) = 0.00085.$$

That is, out of 10,000 coin tosses, transmission at this rate would introduce distortion in (0.00085) (10,000) = 8.5 bits.

□

The encrypter hides or scrambles information so that unintended listeners are unable to discern the information content. The codes used for encryption are generally different from the codes used for error correction.

Encryption is often what the layperson frequently thinks of when they think of "codes," as in secret codes, but as we are seeing, there are many other different kinds of codes.

As we will see, however, the mathematical tools used in error correction coding can be applied to some encryption codes. In particular, we will meet RSA public key encryption as an outgrowth of number theoretic tools to be developed, and McEliece public key encryption as a direct application of a particular family of error correction codes.

The channel coder is the first step in the error correction or error detection process.

The channel coder adds redundant information to the stream of input symbols in a way that allows errors which are introduced into the channel to be corrected. **This book is primarily dedicated to the study of the channel coder and its corresponding channel decoder.**

It may seem peculiar to remove redundancy with the source encoder, then turn right around and add redundancy back in with the channel encoder. However, the redundancy in the source typically depends on the source in an unstructured way and may not provide uniform protection to all the information in the stream, nor provide any indication of how errors occurred or how to correct them. The redundancy provided by the channel coder, on the other hand, is introduced in a structured way, precisely to provide error control capability.

Treating the problems of data compression and error correction separately, rather than seeking a jointly optimal source/channel coding solution, is asymptotically optimal (as the block sizes get large). This fact is called the *source–channel separation theorem* of information theory. (There has been work on combined source/channel coding for finite — practical — block lengths, in which the asymptotic theorems are not invoked. This work falls outside the scope of this book.)

Because of the redundancy introduced by the channel coder, there must be more symbols at the output of the coder than at the input. Frequently, a channel coder operates by accepting a block of k input symbols and producing at its output a block of n symbols, with $n > k$. The **rate** of such a channel coder is

$$R = k/n,$$

so that $R < 1$.

The input to the channel coder is referred to as the *message symbols* (or, in the case of binary codes, the *message bits*). The input may also be referred to as the *information symbols* (or bits).

The modulator converts symbol sequences from the channel encoders into signals appropriate for transmission over the channel. Many channels require that the signals be sent as a continuous-time voltage, or an electromagnetic waveform in a specified frequency band. The modulator provides the appropriate channel-conforming representation.

Included within the modulator block one may find codes as well. Some channels (such as magnetic recording channels) have constraints on the maximum permissible length of runs of 1s. Or they might have a restriction that the sequence must be DC-free. Enforcing such constraints employs special codes. Treatment of such runlength-limited codes appears in [275]; see also [209].

Some modulators employ mechanisms to ensure that the signal occupies a broad bandwidth. This *spread-spectrum* modulation can serve to provide multiple-user access, greater resilience to jamming, low probability of detection, and other advantages. (See, e.g., [509].) Spread-spectrum systems frequently make use of pseudorandom sequences, some of which are produced using linear feedback shift registers as discussed in Appendix 4.A.

The channel is the medium over which the information is conveyed. Examples of channels are telephone lines, internet cables, fiber-optic lines, microwave radio channels, high-frequency channels, cell phone channels, etc. These are channels in which information is conveyed between two distinct places. Information may also be conveyed between two distinct times, for example, by writing information onto a computer disk, then retrieving it at a later time. Hard drives, CD-ROMs, DVDs, and solid-state memory are other examples of channels.

As signals travel through a channel, they are corrupted. For example, a signal may have noise added to it; it may experience time delay or timing jitter, or suffer from attenuation due to propagation distance and/or carrier offset; it may be reflected by multiple objects in its path, resulting in constructive and/or destructive interference patterns; it may experience inadvertent interference from other channels, or be deliberately jammed. It may be filtered by the channel response, resulting in interference among symbols. These sources of corruption in many cases can all occur simultaneously.

For purposes of analysis, channels are frequently characterized by mathematical models, which (it is hoped) are sufficiently accurate to be representative of the attributes of the actual channel, yet are also sufficiently abstracted to yield tractable mathematics. Most of our work in this book will assume one of two idealized channel models, the binary symmetric channel (BSC) and the additive white Gaussian noise channel (AWGNC), which are described in Section 1.5. While these idealized models do not represent all of the possible problems a signal may experience, they form a starting point for many, if not most, of the more comprehensive channel models. The experience gained by studying these simpler channel models forms a foundation for more accurate and complicated channel models. (As exceptions to the AWGN or BSC rule, in Section 14.7, we comment briefly on convolutive channels and turbo equalization, while in Chapter 19, coding for quasi-static Rayleigh flat fading channels are discussed.)

As suggested by Figure 1.2, the channel encoding and modulation may be combined in what is known as *coded modulation*.

Channels have different information-carrying capabilities. For example, a dedicated fiber-optic line is capable of carrying more information than a plain-old-telephone-service (POTS) pair of copper wires. Associated with each channel is a quantity known as the **capacity**, C, which indicates how much information it can carry **reliably**.

The information a channel can reliably carry is intimately related to the use of error correction coding. The governing theorem from information theory is Shannon's **channel coding theorem**, which states essentially this: Provided that the rate R of transmission is less than the capacity C, *there exists* a code such that the probability of error can be made arbitrarily small.

The demodulator/equalizer receives the signal from the channel and converts it into a sequence of symbols. This typically involves many functions, such as filtering, demodulation, carrier synchronization, symbol timing estimation, frame synchronization, and matched filtering, followed